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Radical formula and prime submodules

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Abstract

Let B be a submodule of an R-module M. The intersection of all prime submodules of M containing B is denoted by $\operatorname{rad}(B)$. For every positive integer n, a generalization of $E(B)$ denoted by $E_n(B)$ of M will be introduced. Moreover, $\langle E(B)\rangle \subseteq \langle E_n(B)\rangle \subseteq \operatorname{rad}(B)$. In this paper we will study the equality $\langle E_n(B)\rangle = \operatorname{rad}(B)$. It is proved that if R is an arithmetical ring of finite Krull dimension n, then $\langle E_n(B)\rangle = \operatorname{rad}(B)$.

Keywords: Arithmetical rings; Prime submodules; Radical formula

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring, M a unitary R-module and $\mathbb{N}$ the set of positive numbers.

Let N be a proper submodule of M. It is said that N is a prime submodule of M, if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M) = \{t \in R \mid tM \subseteq N\}$, we say that N is a P-prime submodule of M, and it is easy to see that P is a prime ideal of R. Prime submodules have been studied in several papers such as [1–6,9–14,17].

Recall that for an ideal I of a ring R, the radical of I denoted by $\sqrt{I}$ is defined to be $\sqrt{I} = \{r \in R \mid r^n \in I, \text{ for some } n \in \mathbb{N}\}$.

Also for any subset B of M, the envelope of B, $E(B)$ is defined to be:

$$E(B) = \{x \mid x = ra,\ r^n a \in B,\ \text{for some } r \in R,\ a \in M,\ n \in \mathbb{N}\}.$$

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$\langle E(B)\rangle$ is a module version of the radical of ideals and evidently $B \subseteq \langle E(B)\rangle$.

We know that for an ideal I of R we have, $\sqrt{\sqrt{I}} = \sqrt{I}$. So in studying the radical of an ideal the number of radicals are not important. But for its generalization to modules, i.e., $\langle E(B)\rangle$, it is not correct. For an example see the following.

Example 1. Consider $R = \mathbb{Z}[X]$ and let the R-module M to be $M = R \oplus R$ and $N = \{(r, s) \in M \mid 4r - sX \in RX^2\}$. Then according to [17, p. 110], for the submodule N we have, $\langle E(N)\rangle = R(0, 4) + XM \neq R(0, 2) + XM$. Hence $(0, 2) \notin \langle E(N)\rangle$, however $2^2(0, 1) = (0, 4) \in \langle E(N)\rangle$ and consequently $(0, 2) = 2(0, 1) \in E(\langle E(N)\rangle)$. Thus $\langle E(N)\rangle \neq \langle E(\langle E(N)\rangle)\rangle$.

On the other hand, if $\langle E(B)\rangle$ is a module version of the radical of ideals, so are $\langle E(\langle E(B)\rangle)\rangle$, $\langle E(\langle E(\langle E(B)\rangle)\rangle)\rangle$ and so on.

This discussion leads us to consider a generalization of $E(B)$ in the following definition.

Definition. For a submodule B of M, we will define $E_0(B) = B$, $E_1(B) = E(B)$, $E_2(B) = E(\langle E(B)\rangle)$ and for any positive number n, it will be defined $E_{n+1}(B) = E(\langle E_n(B)\rangle)$ inductively. We will call $E_n(B)$ the nth-envelope of B.

Recall that for an ideal I of R we have, $\sqrt{I} = \bigcap_{P \text{ prime ideal, } I \subseteq P} P$ (see [15, p. 3]).

Let B be a proper submodule of M. The intersection of all prime submodules of M containing B is denoted by $\mathrm{rad}(B)$. If there does not exist any prime submodule of M containing B, then we say $\mathrm{rad}(B) = M$. It is said that M satisfies the radical formula, if for every submodule B of M, $\langle E(B)\rangle = \mathrm{rad}(B)$. It is said that a ring R satisfies the radical formula, if every R-module satisfies the radical formula (see, for example, [4,6,11,13,14,16,17]).

In [13], the authors characterize all commutative Noetherian rings which satisfy the radical formula. In particular it is proved that a commutative Noetherian domain satisfies the radical formula if and only if it is a Dedekind domain. Thus it is a natural question whether Prüfer domains satisfy the radical formula.

Recall that a ring R is said to be an arithmetical ring, if for all ideals I, J and K of R we have, $I + (J \cap K) = (I + J) \cap (I + K)$ (see [7,8]). Obviously Prüfer domains and Dedekind domains are arithmetical.

In this paper we will prove that every arithmetical ring R with $\dim R \leqslant 1$ satisfies the radical formula.

Definition. Let n be a non-negative number. If $\langle E_n(B)\rangle = \mathrm{rad}(B)$, for every submodule B of M, we will say that M satisfies the radical formula of degree n. It will be said that the ring R satisfies the radical formula of degree n, if every R-module satisfies the radical formula of degree n.

We will show that every arithmetical ring of finite Krull dimension n satisfies the radical formula of degree n.

2. Radical formula

Lemma 2.1. *Let B be a submodule of an R-module M, S a multiplicatively closed subset of R and n a non-negative number.*

(i) *If $x \in E_n(B)$, then $Rx \subseteq E_n(B)$.*

(ii) $(\langle E_n(B)\rangle)_S = \langle E_n(B_S)\rangle$.
(iii) $B \subseteq \langle E(B)\rangle \subseteq \langle E_2(B)\rangle \subseteq \langle E_3(B)\rangle \subseteq \cdots \subseteq \operatorname{rad}(B)$.
(iv) $(\bigcup_{m\in\mathbb{N}}\langle E_m(B)\rangle)_S = \bigcup_{m\in\mathbb{N}}\langle E_m(B_S)\rangle$.
(v) $(\operatorname{rad}(B))_S \subseteq \operatorname{rad}(B_S)$.
(vi) *If for every maximal ideal p of R and every R_p-module,* $\operatorname{rad}(0) = \langle E_n(0)\rangle$, *then the ring R satisfies the radical formula of degree n.*

Proof. The proof is easy and is left to the reader. $\square$

Example 2. In Example 1, we have

$$N \subset \langle E(N)\rangle = R(0,4) + XM \subset R(0,2) + XM = \langle E_2(N)\rangle = \operatorname{rad}(N).$$

Proof. By [17, p. 110], we have, $\langle E(N)\rangle = N + XM = R(0,4) + XM$ and $\operatorname{rad}(N) = R(0,2) + XM$. In Example 1, we showed that $(0,2) \in E_2(N)$. So $R(0,2) \subseteq \langle E_2(N)\rangle$. By Lemma 2.1(iii), we have, $N \subseteq \langle E_n(N)\rangle \subseteq \operatorname{rad}(N)$, for every $n \in \mathbb{N}$. Hence $N \subset \langle E(N)\rangle = R(0,4) + XM \subset R(0,2) + XM \subseteq \langle E_2(N)\rangle \subseteq \operatorname{rad}(N) = R(0,2) + XM$. Therefore $N \subset \langle E(N)\rangle = R(0,4) + XM \subset R(0,2) + XM = \langle E_2(N)\rangle = \operatorname{rad}(N)$. $\square$

Lemma 2.2. *A ring R is arithmetical if and only if for each prime ideal p of R, every two ideals of the ring R_p are comparable.*

Proof. See [7, Theorem 1]. $\square$

Lemma 2.3. *Let R be a local arithmetical ring, r a non-unit element of R, and $I = \bigcap_{n\in\mathbb{N}} Rr^n$. Then*

(i) *If for some $k \in \mathbb{N}$, $r^k \in I$, then $r^k = 0$.*
(ii) *I is a prime ideal of R, or r is a nilpotent element of R.*

Proof. (i) Note that $r^k \in I \subseteq Rr^{k+1}$. So for some $c \in R$ we have, $r^k = cr^{k+1}$, that is, $r^k(1 - cr) = 0$. Since $1 - cr$ is a unit, $r^k = 0$.

(ii) Let $ab \in I = \bigcap_{n\in\mathbb{N}} Rr^n$, where $a, b \in R$, $a \notin I$ and $b \notin I$. Then $a \notin Rr^n$ and $b \notin Rr^m$ for some $n, m \in \mathbb{N}$. By Lemma 2.2, every two ideals of R are comparable, so $Rr^n \subseteq Ra$ and $Rr^m \subseteq Rb$. Thus $r^{n+m} \in Rr^{n+m} \subseteq Rab \subseteq I$. Consequently by part (i), $r^{n+m} = 0$. $\square$

Theorem 2.4.

(i) *Every arithmetical ring of finite Krull dimension k satisfies the radical formula of degree k.*
(ii) *Every arithmetical ring R with* $\dim R \leqslant 1$ *satisfies the radical formula.*

Proof. (i) Consider $E^0(0) = 0$, $E^1(0) = E(0)$, $E^2(0) = E(E(0))$ and for every $n \in \mathbb{N}$, define $E^{n+1}(0) = E(E^n(0))$ inductively. It is easy to see that for every $\delta \in \mathbb{N}$ we have

$$\begin{aligned} E^\delta(0) = \bigl\{x \mid \text{for } i = 1, 2, \ldots, \delta,\ \exists r_i \in R,\ \exists x_i \in M,\ \exists n_i \in \mathbb{N},\ \ni x = r_1x_1, \\ r_1^{n_1}x_1 = r_2x_2,\ r_2^{n_2}x_2 = r_3x_3, \ldots, r_{\delta-1}^{n_{\delta-1}}x_{\delta-1} = r_\delta x_\delta,\ r_\delta^{n_\delta}x_\delta = 0\bigr\}. \end{aligned} \quad (*)$$

Let R be an arithmetical ring and $\dim R = k$. By Lemma 2.1(vi), it is enough to show that for every maximal ideal p of R, for every R_P-module we have, $\mathrm{rad}(0) = \langle E_k(0)\rangle$. Indeed we will show that $\mathrm{rad}(0) \subseteq E^k(0)$, and it is easy to see that $E^k(0) \subseteq E_k(0) \subseteq \langle E_k(0)\rangle$. Also by Lemma 2.1(iii), $\langle E_k(0)\rangle \subseteq \mathrm{rad}(0)$. Therefore we will have $E^k(0) = E_k(0) = \langle E_k(0)\rangle = \mathrm{rad}(0)$.

By localization and Lemma 2.2, we may assume that R is a local ring with a maximal ideal m such that every two ideals of R are comparable. Let M be an R-module. Consider $x \in \mathrm{rad}(0)$. We will prove that $x \in E^k(0)$.

Since m is a maximal ideal of R, mM is a prime submodule of M or $mM = M$. Thus $x \in mM$. Then $x = \sum_{i=1}^{l} r_i a_i$ such that for each i, $1 \leqslant i \leqslant l$, $r_i \in m$ and $a_i \in M$. Every two ideals of R are comparable, then $\{Rr_i,\ i = 1, 2, 3, \dots, l\}$ is a chain of ideals of R. Without loss of generality we may suppose that Rr_1 is the maximal element of this chain. So $x = r_1 x_1$, for some $x_1 \in M$. Let $S_1 = \{r_1^n x_1 \mid n \in \mathbb{N}\}$. Now define the set T_1 as follows:

$$T_1 = \left\{K \;\middle|\; K \text{ is a submodule of } M,\ K \cap S_1 = \emptyset,\ \bigcap_{n\in\mathbb{N}} Rr_1^n \subseteq (K : M)\right\}.$$

First we will show that $T_1 = \emptyset$. If $T_1 \neq \emptyset$, then by Zorn's Lemma, T_1 has a maximal element. Let N be a maximal element of T_1. We show that N is a prime submodule of M. Suppose $ay \in N$, where $a \in R$ and $y \in M$. We have one of the following.

(i) $Ra \subseteq Rr_1^n$, for every $n \in \mathbb{N}$.
(ii) $Ra \nsubseteq Rr_1^d$, for some $d \in \mathbb{N}$.

If (i) holds, then $a \in Ra \subseteq \bigcap_{n\in\mathbb{N}} Rr_1^n \subseteq (N : M)$, so we have the proof.

If (ii) is satisfied, since every two ideals of R are comparable, we have, $Rr_1^d \subseteq Ra$. Let $r_1^d = ba$, where $b \in R$. If $y \notin N$, then $\bigcap_{n\in\mathbb{N}} Rr_1^n \subseteq (N : M) \subseteq (N + Ry : M)$ and N is a maximal element of T_1, then $(N + Ry) \cap S_1 \neq \emptyset$. Consider $r_1^t x_1 \in (N + Ry) \cap S_1$, where $t \in \mathbb{N}$. Then $r_1^t x_1 = n' + cy$, where $n' \in N$ and $c \in R$. Now $r_1^{d+t} x_1 = r_1^t bax_1 = ban' + cbay \in N$, that is $N \cap S_1 \neq \emptyset$, which is a contradiction. So $y \in N$. Therefore N is a prime submodule of M and since $N \in T_1$ we have, $N \cap S_1 = \emptyset$, which is a contradiction with the fact that $r_1 x_1 = x \in S_1 \cap \mathrm{rad}(0) \subseteq S_1 \cap N$. Consequently $T_1 = \emptyset$.

Put $\bigcap_{n\in\mathbb{N}} Rr_1^n = I_1$. By Lemma 2.3(ii), r_1 is a nilpotent element or I_1 is a prime ideal of R. If r_1 is nilpotent and $r_1^\alpha = 0$, for some $\alpha \in \mathbb{N}$, then $r_1^\alpha x_1 = 0$. So $x = r_1 x_1 \in E(0) \subseteq E^k(0)$. Now assume that I_1 is a prime ideal of R. Note that $T_1 = \emptyset$, then $I_1 M \notin T_1$ and since $I_1 \subseteq (I_1 M : M)$ we have, $I_1 M \cap S_1 \neq \emptyset$. Then let $r_1^{n_1} x_1 = \sum_{j=1}^{h} i_j m_j$, where $n_1 \in \mathbb{N}$, $i_j \in I_1$, $m_j \in M$, for each $1 \leqslant j \leqslant h$. Note that $\{Ri_j,\ j = 1, 2, 3, \dots, h\}$ is a chain of ideals of R, then we may assume that $r_1^{n_1} x_1 = r_2 x_2$, where $r_2 \in I_1$ and $x_2 \in M$.

Now consider $S_2 = \{r_2^n x_2 \mid n \in \mathbb{N}\}$, and define the set T_2 as follows:

$$T_2 = \left\{K \;\middle|\; K \text{ is a submodule of } M,\ K \cap S_2 = \emptyset,\ \bigcap_{n\in\mathbb{N}} Rr_2^n \subseteq (K : M)\right\}.$$

A similar proof to that of above will show that T_2 is an empty set and r_2 is a nilpotent element or $I_2 = \bigcap_{n\in\mathbb{N}} Rr_2^n$ is a prime ideal of R. If r_2 is nilpotent, then for some positive number n_2 we have, $r_2^{n_2} = 0$. Therefor we have, $x = r_1 x_1$, $r_1^{n_1} x_1 = r_2 x_2$, $r_2^{n_2} x_2 = 0$. So by $(*)$ we have, $x \in E^2(0) \subseteq E^k(0)$. Now suppose that r_2 is not a nilpotent element. Then I_2 is a prime ideal of R. Since $r_2 \in I_1$, $I_2 \subseteq I_1$. If $I_2 = I_1$, then $r_2 \in I_1 = I_2 = \bigcap_{n\in\mathbb{N}} Rr_2^n$. Now by Lemma 2.3(i), $r_2 = 0$. So r_2 is a nilpotent element, which is a contradiction. Hence $I_2 \subset I_1$. Also note that $I_1 \subset m$, otherwise $r_1 \in m = I_1$. Thus by Lemma 2.3(i), $r_1 = 0$. Then r_1 is a nilpotent element, which is a contradiction.

By continuing this argument we will get elements $r_1, r_2, \ldots \in R$, $x_1, x_2, \ldots \in M$ and $n_1, n_2, \ldots \in \mathbb{N}$ such that $r_j^{n_j} x_j = r_{j+1} x_{j+1}$ for each $j \in \mathbb{N}$, and if $I_j = \bigcap_{n\in\mathbb{N}} Rr_j^n$, then $\cdots \subset I_3 \subset I_2 \subset I_1 \subset m$ is a chain of prime ideals of R. Moreover, $r_{j+1} \in I_j$, for each j. Since $\dim R = k$, we have, $I_{k+1} = I_k$. Consequently $r_{k+1} \in I_k = I_{k+1} = \bigcap_{n\in\mathbb{N}} Rr_{k+1}^n$. Again by Lemma 2.3(i), $r_{k+1} = 0$, that is, $r_k^{n_k} x_k = r_{k+1} x_{k+1} = 0$. Now we have

$$x = r_1 x_1, \qquad r_1^{n_1} x_1 = r_2 x_2, \qquad r_2^{n_2} x_2 = r_3 x_3, \qquad \ldots, \qquad r_{k-1}^{n_{k-1}} x_{k-1} = r_k x_k, \qquad r_k^{n_k} x_k = 0.$$

Consequently by $(*)$ we have, $x = r_1 x_1 \in E^k(0)$ and the proof is completed.

(ii) If $\dim R \leqslant 1$, then by part (i), for every submodule B of an R-module M we have, $\langle E(B)\rangle = \langle E_1(B)\rangle = \mathrm{rad}(B)$. □

Corollary 2.5. *Let R be an arithmetical ring with DCC on prime ideals. Then for every submodule B of an R-module M, $\varinjlim \langle E_n(B)\rangle = \mathrm{rad}(B)$.*

Proof. Follow the proof of Theorem 2.4. □

Lemma 2.6. *Let M be an R-module with DCC on cyclic submodules, B a submodule of M, and S a multiplicatively closed subset of R.*

(i) *M/B as an R-module has DCC on cyclic submodules.*
(ii) *M_S as an R_S-module has DCC on cyclic submodules.*

Proof. (i) Let $\cdots \subseteq R(x_3 + B) \subseteq R(x_2 + B) \subseteq R(x_1 + B)$ be a descending chain of cyclic submodules of M/B, where $x_1, x_2, x_3, \ldots \in M$. Since $R(x_2 + B) \subseteq R(x_1 + B)$, there exist $r_2 \in R$ and $b_2 \in B$ such that $x_2 = r_1 x_1 + b_2$. Now since $R(x_3 + B) \subseteq R(x_2 + B) = R(x_2 - b_2 + B)$, there exist $r_3 \in R$ and $b_3 \in B$ such that $x_3 = r_3(x_2 - b_2) + b_3$. By continuing this process, for each $1 < n \in \mathbb{N}$, we will get $b_{n+1} \in B$ and $r_{n+1} \in R$ such that $x_{n+1} = r_{n+1}(x_n - b_n) + b_{n+1}$. Consequently the following is a descending chain of cyclic submodules of M:

$$\cdots \subseteq R(x_3 - b_3) \subseteq R(x_2 - b_2) \subseteq Rx_1.$$

Hence by our assumption there exists $m \in \mathbb{N}$ such that for each $k > m$ we have, $R(x_k - b_k) = R(x_m - b_m)$, which implies that $R(x_k + B) = R(x_m + B)$.

(ii) Obviously every cyclic submodule of M_S is of the form $R_S \frac{x}{1}$ where $x \in M$. Then let $\cdots \subseteq R_S \frac{x_3}{1} \subseteq R_S \frac{x_2}{1} \subseteq R_S \frac{x_1}{1}$ be a descending chain of cyclic submodules of M_S, where $x_1, x_2, x_3, \ldots \in M$.

Since $R_S \frac{x_2}{1} \subseteq R_S \frac{x_1}{1}$, we have, $\frac{x_2}{1} = \frac{r_1 x_1}{s_1}$, for some $r_1 \in R$ and $s_1 \in S$. Then there exists $s_2 \in S$ such that $s_2 s_1 x_2 = s_2 r_1 x_1$. Put $t_2 = s_2 s_1$. Then $t_2 \in S$ and we have, $Rt_2 x_2 \subseteq Rx_1$. Now since $R_S \frac{x_3}{1} \subseteq R_S \frac{x_2}{1} = R_S \frac{t_2 x_2}{1}$, similarly there exists $t_3 \in S$ such that $Rt_3 x_3 \subseteq Rt_2 x_2$.

By continuing this process, for each $1 < n \in \mathbb{N}$, we will get $t_{n+1} \in S$ such that $Rt_{n+1} x_{n+1} \subseteq Rt_n x_n$. Consequently the following is a descending chain of cyclic submodules of M, $\cdots \subseteq$

$Rt_3x_3 \subseteq Rt_2x_2 \subseteq Rx_1$. Hence by our assumption there exists $m \in \mathbb{N}$ such that for each $k > m$ we have, $Rt_kx_k = Rt_mx_m$, which implies that $R_S\frac{x_k}{1} = R_S\frac{x_m}{1}$. □

Corollary 2.7. *Let R be an arithmetical ring, and M an R-module with DCC on cyclic submodules. Then for every submodule B of M, $\varinjlim\langle E_n(B)\rangle = \mathrm{rad}(B)$.*

Proof. By Lemma 2.6(i), M/B as an R-module has DCC on cyclic submodules. Now by Lemma 2.6(ii), every localization of M/B has DCC on cyclic submodules. So by localization we may assume that R is a local arithmetical ring such that every two ideals of R are comparable, and it is enough to show that $\bigcup_{n\in\mathbb{N}}\langle E_n(0)\rangle = \mathrm{rad}(0)$.

Let $x \in \mathrm{rad}(0)$. By following the proof of Theorem 2.4, we will get a sequence $\{r_jx_j\}_{j\in\mathbb{N}}$ of elements of M such that for each j, $r_j^{n_j}x_j = r_{j+1}x_{j+1}$. Since the chain $\cdots \subseteq Rr_3x_3 \subseteq Rr_2x_2 \subseteq Rr_1x_1$ stops, for some $k \in \mathbb{N}$ we have, $Rr_kx_k = Rr_{k+1}x_{k+1} = Rr_k^{n_k}x_k$. Therefore there exists $v \in R$ such that $r_k(1 - r_k^{n_k-1}v)x_k = 0$. Note that $r_k \in m$, then $1 - r_k^{n_k-1}v$ is a unit. So $r_{k-1}^{n_{k-1}}x_{k-1} = r_kx_k = 0$. Thus by $(*)$ in Theorem 2.4, we have, $x = r_1x_1 \in E^{k-1}(0) \subseteq E_{k-1}(0) \subseteq \bigcup_{n\in\mathbb{N}}\langle E_n(0)\rangle$. Therefore $\bigcup_{n\in\mathbb{N}}\langle E_n(0)\rangle = \mathrm{rad}(0)$. □

Remark. Let B be a submodule of an R-module M.

(i) If M satisfies the radical formula of any degree we have, $\bigcup_{n\in\mathbb{N}}\langle E_n(B)\rangle = \mathrm{rad}(B)$, by Lemma 2.1(iii). Now suppose that R is an arithmetical ring. According to Theorem 2.4, Corollaries 2.5 and 2.7, $\bigcup_{n\in\mathbb{N}}\langle E_n(B)\rangle = \mathrm{rad}(B)$, if one of the following holds:
   (a) $\dim R < \infty$.
   (b) R has DCC on prime ideals.
   (c) M has DCC on cyclic submodules.
   These facts inspire us to define the radical envelope of a submodule B to be $\bigcup_{n\in\mathbb{N}}\langle E_n(B)\rangle$ and not just $\langle E(B)\rangle$.

(ii) In general $E_k(B)$ is not necessarily a submodule of M. Now suppose that R is a local arithmetical ring of finite Krull dimension k. According to the proof of Theorem 2.4, we have, $E^k(B) = E_k(B) = \mathrm{rad}(B)$. Consequently $E^k(B) = E_k(B)$ and it is a submodule of M. Moreover, $\langle E^k(B)\rangle = \langle E_k(B)\rangle = \bigcup_{n\in\mathbb{N}}\langle E_n(B)\rangle = \mathrm{rad}(B)$.

(iii) Let R be an arithmetical ring of finite Krull dimension k. In Lemma 2.1, we may replace $E_n(B)$ with $E^n(B)$. Now by the new version of this lemma and part (ii) of this remark, we have $(\langle E^k(B)\rangle)_P = \langle E^k(B_P)\rangle = \langle E_k(B_P)\rangle = (\langle E_k(B)\rangle)_P$, for each prime ideal P of R. Consequently $\langle E^k(B)\rangle = \langle E_k(B)\rangle = \mathrm{rad}(B)$.

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